chapter 7

The Habit of Clinging to an Ultimate Ground[1]

Every scientific 'fulfillment' raises new 'questions'; it asks *to be 'surpassed' and outdated. . . . In principle, this progress goes on* ad infinitum. *And with this we come to inquire into the* meaning *of science. . . . Why does one engage in doing something that in reality never comes, and never can come, to an end?*

—Weber, *Science as a Vocation*

The notions of real interest to mathematicians like myself are not on the printed page. They lurk behind the doors of conception. It is believed[2] that they will some day emerge and shed so much light on earlier concepts that the latter will disintegrate into marginalia. By their very nature, they elude precise definition, so that on the conventional account they are scarcely mathematical at all. Coming to grips with them is not to be compared with attempting to solve an intractable problem, an experience that drives most narratives of mathematical discovery. What I have in mind harbors a more fundamental obscurity. One cannot even formulate a problem, much less attempt to solve it; the items (notions, concepts) in terms of which the problem would be formulated have yet to be invented. How can we talk to one another, or to ourselves, about the mathematics we were born too soon to understand?

Maybe with the help of mind-altering drugs? That's what you might conclude from the snippet of dialogue from *Proof* quoted in the last chapter, but you would be wrong. It seems *Proof*'s author got that idea by reading Paul Hoffman's biography of Paul Erdős, the Hungarian mathematician famous for finding unexpectedly rich structures in apparently elementary mathematics; for spending nearly all his life wandering

across the planet, crashing in the homes of mathematicians and inviting them to join him in solving problems, sometimes in exchange for small monetary rewards; and for saying "a mathematician is a machine for transforming coffee into theorems."[3] Heraclitus considered most people "oblivious of what they do when awake, just as they are forgetful of what they do asleep."[4] Hoffman depicts Erdős, in contrast, as the Dr. Gonzo of mathematics: "for the last twenty-five years of his life. . . [Erdős] put in nineteen-hour days . . . fortified with 10 to 20 milligrams of Benzedrine or Ritalin, strong espresso, and caffeine tablets," with the result that he spent the last quarter of the twentieth century fully focused on the kind of mathematics at everyone's disposal, seeing things there that perhaps he was the only one awake enough to see.[5]

Grothendieck, by his own account, spent much of his life dreaming his way across the same landscape, forcing a path into a conceptual future that ceaselessly receded, riding not a drug habit but rather a refined mathematical minimalism. "He seemed to have the knack," wrote number theorist John Tate, "time after time, of stripping away just enough. . . . It's streamlined; there is no baggage. It's just right."[6] Grothendieck's search for what his colleague Roger Godement once called "total purity" encompassed his physical existence as well: in São Paolo in 1953–1954, he subsisted on milk and bananas, and years after his withdrawal from mathematics, during a forty-five-day fast in 1990, he went so far as to refuse to drink liquids, bringing himself to the point of death in an apparent attempt to "compel God . . . to reveal himself."[7] And yet what he was really seeking, in life as in mathematics, continued to elude his grasp.

Incarnation

Sometimes the curtain can be persuaded to part slightly and the elusive items can be approached with the help of *punctuation*—quotation marks, for example, which mathematicians use frequently, especially in expository writing, in at least four distinct (though overlapping) ways:

1. For direct quotations (the way everyone uses them).
2. For implicit quotations (of something that is often said in the field, as in the first two examples in item 3).

3. To substitute for the cumbersome procedure of formal definition, in other words as a synonym for "so-called." Thus Eric Zaslow writes:[8]

> We call such a term a "correlation function."
>
> Such a case is known as an "anomaly."

and

> The quantum Hilbert space is then a (tensor) of lots of different "occupation number Hilbert spaces."

The last example is more complex than the first two—visibly, Zaslow is making it clear to the reader that he has in mind a notion deserving a formal definition, but that it would be distracting to present it here and unnecessary as well, since the reader can probably figure out what's intended.

But the really interesting use of quotation marks is

4. When an analogy that is, strictly speaking, *incorrect* offers a better description of a notion—a better fit with intuition—than its formal definition. For example,

> you can think of actions as "molecules" and transitive actions as the "atoms" into which they can be decomposed.[9]

Since the sentence has exactly the same meaning without the quotation marks, one has to assume that they have been inserted to stress that the invitation to (chemical) intuition is also explicitly an invitation to relax one's critical sense. Some seminar speakers write such quotation marks on the board; others make the quotation-mark gesture, wiggling two fingers on each hand while uttering the problematic word; still others preface an informal explanation by something like *Let me explain this in words*,[10] which is literally meaningless unless understood as a warning that formal syntactical rules are temporarily suspended, to be replaced by some other kind of mathematics.

More piquant examples can be found in situations where normal semantics do not suffice for communication among specialists. Since I remember that it was advisable, before the 1991 two-week workshop on *Motives* in Seattle, to use the term *motive* only between scare quotes—I did so myself—I am gratified to see that Grothendieck made repeated

use of type 4 quotation marks when explaining why he introduced the notion, in his unpublished 1986 manuscript *Récoltes et Sémailles* (*ReS*):

> One has the distinct impression (but in a sense that remains vague) that each of these [cohomological] theories "amount to the same thing," that they "give the same results." In order to express this intuition . . . I formulated the notion of "*motive*" associated to an algebraic variety. By this term, I want to suggest that it is the "common motive" (or "common *reason*") behind this multitude of cohomological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possible.[11]

Cohomological, or its noun form *cohomology*, is the technical name for a method, properly belonging to topology, that introduces an *algebraic structure* in an attempt to get at the "essence" (a word some philosophers write in scare quotes) of a shape. So if the topological essence of the infinity sign (see figure 7.1) is that it consists of two attached holes, cohomology is a way of doing arithmetic (addition, subtraction, multiplication) with these holes. For example

$$4(\text{left holes}) - 5(\text{right holes})$$

is a legitimate formula in the cohomology of figure 7.1, which is also an (unorthodox) picture of the *cubic* equation $y^2 = x^2 (x - 1)$ (see chapters 2 and γ), whose holes are, therefore, meaningful in number theory.

Structure, already encountered in the second paragraph, is a loaded term. Historians stress the Bourbaki group's role in making structures central to their mathematical architecture, following the structural revolution in algebra undertaken in the 1920s and 1930s by Emmy Noether and her protégé Bartel Van der Waerden.[12] Grothendieck was an active Bourbakiste during the late 1950s, and the word *structure* occurs on 124 of the 929 pages of *ReS*. On page 48, for example: "among the thousand-and-one faces form chooses to reveal itself to us, the one that has fascinated me more than any other and continues to fascinate me is the **structure** hidden in mathematical things" (my translation; emphasis in the original). Weil was a founder of Bourbaki; his insight mentioned in chapter 2, the one that converted me to number theory, was that the geometric essence of a problem in number theory could be grasped by

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Figure 7.1. John Wallis's symbol for infinity.

means of the algebraic and topological structure of cohomology. And the *motive* Grothendieck hoped would provide the proper setting for Weil's conjectures is a structure that clenches this essence even more tightly. Grothendieck explains the choice of term by musical analogies:

> *Ces différentes théories cohomologiques seraient comme autant de développements thématiques différents, chacun dans le "tempo", dans la "clef" et dans le "mode" ("majeur" ou "mineur") qui lui est propre, d'un même "motif de base" (appelé "théorie cohomologique motivique"), lequel serait en même temps la plus fondamentale, ou la plus "fine", de toutes ces "incarnations" thématiques différentes (c'est-à-dire, de toutes ces théories cohomologiques possibles*).*

Note the word *incarnation*. A philosopher might understand this on the model of an example or instance that points to an "essence" whose place in Grothendieck's text is occuped by the "common motive." On the same page, and several more times over the course of his rambling, lyrical—and often irascible—manuscript, Grothendieck uses the term *avatar* in scare quotes, possibly for the same reason:

> Inspired by certain ideas of Serre, and also by the wish to find a certain common "principle" or "motif" for the various purely algebraic "avatars" that were known, or expected, for the classical Betti cohomology of a complex algebraic variety, I had introduced towards the beginning of the 60s the notion of "motif". [*En m'inspirant de certaines idées de Serre, et du désir aussi de trouver un certain "principe" (ou "motif") commun pour les divers "avatars" purement algébriques connus (ou pressentis) pour la cohomologie de Betti classique d'une variété algébrique complexe, j'avais introduit vers les débuts des années soixante la notion de "motif"*.]

* These different cohomological theories would be like so many different thematic developments, each in its own "tempo," "key," and ("major" or "minor") "mode," of the same "basic motif" (called a "**motivic** cohomology theory"), which would at the same time be the most fundamental, or the "finest," of all these different thematic "incarnations" . . . (that is, of all these possible cohomological theories) (Grothendieck, *ReS*, p. 60).

Motives still attract quotation marks of this type. In 2010 Drinfel'd could write that

> [m]orally, $K_{mot}(X, \mathbf{Q})$ should be the Grothendieck group of the "category of motivic **Q**-sheaves" on X.[13]

(Note the philosophically fraught word *category*, which has a precise meaning in mathematics; we'll be seeing it again.) Drinfel'd adds parenthetically that "the words in quotation marks do not refer to any precise notion of motivic sheaf."

Placing quotation marks around motif or motivic sheaf—or for that matter using the word *morally*, as mathematicians often do, as a functional equivalent of scare quotes—permits invocation of the object one does not know how to define in much the same way that the name-worshipping Moscow set theorists brought sets into being by giving them names. But readers of the French mathematics of a certain generation, especially that associated with the Bourbaki group, can't help but notice that a taste for Indian (rather than, say, Russian Orthodox) metaphysics inflected their terminology. Weil referred to "those obscure analogies, those disturbing reflections of one theory on another"—an early avatar of Grothendieck's avatars—writing that "[a]s the *[Bhagavad]-Gita* teaches, one achieves knowledge and indifference at the same time." Pierre Deligne, whose work did much to embolden mathematicians to remove the scare quotes from "motive," explains the use of the word *yoga*, using the visual metaphor of *panorama*:

> In mathematics, there are not only theorems. There are, what we call, "philosophies" or "yogas," which remain vague. Sometimes we can guess the flavor of what should be true but cannot make a precise statement. . . . A philosophy creates a panorama where you can put things in place and understand that if you do something here, you can make progress somewhere else. That is how things begin to fit together.[14]

"The sad truth," Grothendieck wrote in a letter to Serre in 1964, was that he didn't yet know how to define the [abelian category of] motives he had introduced earlier in the same letter, but he was "beginning to have a rather precise yoga" about these nebulous objects of his imagination. One might argue that Grothendieck in fact made precise definitions and formulated equally precise conjectures that would retrospectively

justify the "distinct . . . but . . . vague" impression of which he wrote, and that the use of the words *yoga* and *avatar*, like the quotation marks, is a theatrical effect that serves merely to make the impression more vivid. One might, with equal justification, ask why just these rhetorical techniques do, in fact, enhance vividness. Working with the various "cohomological theories" of which Grothendieck wrote—just as Deligne did, viewing them as avatars of a more fundamental theory of motives—may leave no logical trace but does represent a different *intentional* relation to the cohomological theories in question:

> A mathematical concept is always a pair of two mutually dependent things: a formal definition on the one hand and an intention on the other hand. He or she who knows the intention of a concept has a kind of "nose" guiding the "right" use of the formal concept.[15]

The author of this quotation, who seems to be using the word *intention* in the way I intended,[16] did well not to lift the veil of (type 4) quotation marks from that which is better left to the reader's imagination. Attempts at greater precision lead straight to the threshold of the abyss of speculative philosophy, where one seeks to explain what it means to take phenomena (impressions, mental images) as symptomatic of something that remains concealed. This applies to intentions as well as to avatars—which is not to say that the (intentional) relation I have in mind is purely subjective. If anything is peculiar about the use of *intentional* in connection with avatars, it is that the concealed "underlying theory" of which the phenomena are supposed to be symptomatic does not yet exist, as if the mathematician's role were to create the source of the shadows they have already seen on the wall of Plato's cave.[17] I limit my speculation to claiming that it matters to mathematicians what they think their work is **about**, ***whether or not it matters to the work***, and that this ought to be a matter of concern for philosophers.

In chapter 3, I suggested that the goal of mathematics is to convert rigorous proofs to heuristics—not to solve a problem, in other words, but rather to reformulate it in a way that makes the solution obvious. As Grothendieck (*ReS*, p. 368) wrote:

> They have completely forgotten what is a **mathematical creation**: a vision that decants little by little over months and years, bringing to light the "ob-

> vious" [*évident*] thing that no one had seen, taking form in an "obvious" assertion of which no one had dreamed . . . and that the first one to come along can then prove in five minutes, using techniques ready to hand [*toutes cuites*].

"Obvious" is the property Wittgenstein called *übersichtlich*, synoptic or perspicuous.[18] This is where the avatars come in. In the situations I have in mind, one may well have a rigorous proof, but the obviousness is based on an understanding that fits only a pattern one cannot yet explain or even define rigorously. The available concepts are interpreted as the *avatars* of the inaccessible concepts we are striving to grasp. So, I agree with Grothendieck and disagree with Wittgenstein when the latter writes (in *On Certainty*):

> the end is not certain propositions' striking us immediately as true, i.e. it is not a kind of *seeing* on our part, it is our *acting*, which lies at the bottom of the language game.

In mathematics this separation of seeing and acting seems artificial; seeing and conveying what one has seen is as important as any other form of acting as a mathematician.

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The word *avatar* was used in English as early as 1784 and in French by 1800 as a form of the Sanskrit word *avatara*, denoting the successive incarnations of the Hindu god Vishnu.[19] By 1815 the word was so familiar that Sir Walter Scott could refer to Napoleon's possible return from Elba as the "third avatar of this singular emanation of the Evil Principle." The meaning relevant to mathematics of "transformation, manifestation, alternative version" first appeared in French in 1822 and in English in 1850.

Que d'avatars dans la vie politique de cet homme! Cette institution va connaître un nouvel avatar (one reads in the *Dictionnaire de l'Académie Française*). Educated French speakers were comfortable using the word in this way in the 1950s, if not earlier. Deligne may have been the first to give it its modern mathematical sense, in a widely read paper from 1971, where he wrote (in French) "*One should consider this as an avatar of the projective system (1.8.1)*." Physicists Sidney Cole-

man, J. David Gross, and Roman Jackiw used the word slightly earlier, and no doubt independently, to roughly the same end.[20] Today, Google Scholar has hundreds of quotations about avatars in geometry—*an algebraic-geometric avatar of higher Teichmüller theory*—or topology—*this operadic cotangent complex will serve as our avatar through much of this work*—or mathematical physics—*topological avatar of the black-hole entropy*—or any other branch of algebra or geometry. The incorporation of the word in the standard lexicon of mathematicians likely has little or no relation to the development of video games or virtual reality experiences like Second Life and can be traced rather to its use by Grothendieck in *Récoltes et Sémailles*, to identify cohomology groups as symptomatic incarnations of the objects of an as-yet inaccessible *category* of motives.

The next time I use the word *category*, I will begin to explain what it means, along with the word *structure*, with which it goes hand in hand, but which I have deliberately left undefined. For the moment, bearing in mind that a motive is a certain kind of algebraic structure, the expression "category of motives" should suggest that such a structure can be grasped only in relation to other structures of the same kind and that the category provides the formal unifying framework in which such relations are made manifest. And just as the introduction of Galois theory brought about a change of perspective, in which the search for solutions of polynomial equations was replaced by a focus on the new structure of the Galois group, you may anticipate that the design of a category of motives meeting Grothendieck's specifications will likely usher in a new mathematical era, in which the motives themselves—not to mention the equations whose "essence" they capture—will no longer be central. Attention will turn instead to the structure of the *category* to which the motives belong,[21] along with other structures to which it can be compared. This is nothing as brutal as a paradigm shift; each generation's new perspective is meant to be more encompassing, as if mathematicians were collectively climbing and simultaneously building a ladder that at each rung offers a broadening panorama and the growing conviction that the process will never end—"knowledge and indifference," as Weil wrote, alluding to the *Gita*; or *ataraxia*, the absence of worry at which Pyrrhonian skepticism aimed in conceding the fruitlessness of the quest for ultimate truth.

chapter 8

The Science of Tricks

Banish the tunes of Cheng and keep clever talkers at a distance. The tunes of Cheng are wanton and clever talkers are dangerous.

—Confucius, *Analects* 15:11[1]

Individuals who never sense the contradictions of their cultural inheritance run the risk of becoming little more than host bodies for stale gestures, metaphors, and received ideas.

—Lewis Hyde, *Trickster Makes This World*, p. 307

My *Kunstgriff*

In January 2010 it was revealed that I am a trickster. Toby Gee, a young English mathematician, broke the story. Before a packed Paris auditorium, Gee explained how he and two even-younger colleagues had found a new way to exploit what he called "Harris's tensor product trick"—implicitly allowing that it might not be the only item in my bag of tricks, that I'm not necessarily a one-trick pony—to improve on my most recent work. I had first employed the trick in a solo paper, but it had been recycled to much greater effect in my joint paper with Richard Taylor and two of his students, who in turn recycled themselves as Toby Gee's collaborators. During his talk, he hinted that there was more to the story. At the break Gee let me know that by March, he and all three of my erstwhile collaborators would have combined a version of my trick with a few new "key ideas" to prove a big new theorem that incorporates most previous results in the field—notably all the results to which my name was attached.

The trickster, in one of many disguises—the Yoruba Esu-Elegbara or his African American equivalents, the Signifying Monkey and Br'er

Rabbit; the Winnebago Wakdjunkaga, and the Coyote of the American Southwest; or Hermes, Prometheus, and Loki from European mythology—attended nearly every skirmish of the 1990s culture wars but had already tired of running laps around the postcolonialist circuit by the time the archetype's presence as a mathematical figure was abruptly brought to my attention. And here I am barely into the second paragraph and already playing tricks on you—at least two, if you're keeping count, with more on the way. Most blatantly, there's no such thing as a trickster in mathematics. There are plenty of mathematical tricks, enough to fill a "Tricks Wiki" or "Tricki," a "repository for mathematical tricks and techniques."[2] But even the rare mathematicians skilled at inventing trick after trick—the "mathematical wit" Paul Erdős comes to mind—are not known as "tricksters." What my colleagues call a trick is a kind of mathematical gesture or speech act; and what I'm calling a trickster is a role, a persona, one of several, an attribute not of the author who invents or employs it but rather of the text in which this kind of argument or idea or style of thought appears.[3]

As for the trick Toby Gee mentioned in his lecture, it wasn't really even mine, although I was the first to notice its relevance in the context of mutual interest. I learned about this particular class of tensor product tricks from a few of my contemporaries (when I was a not-yet-old dog—just around Toby Gee's age as I write this) and I've applied them—we don't "play" tricks, much less "turn" tricks, in mathematics—to solve problems in at least three completely different settings. So it would be ironic[4] if "Harris's tensor product trick" were to be the last of my marks to fade from number theory, like the Cheshire cat's smile. And as "tricky Dick" Nixon's career reminds us, it is a mixed blessing to be remembered primarily for one's tricks.

You don't really want to know the details of my trick, but you may be wondering what makes it a trick rather than some other kind of mathematical gesture. I ought to be able to answer that question if anyone can, because I'm the one who called it a tensor product trick! Whatever was I thinking? Frankly, I wish I knew. To determine when I began to view this kind of argument as a trick would require an experiment in personal intellectual archaeology or at least a time-consuming scouring of my old hard drives; but it's too late to deny my responsibility for the terminology.[5] There's my usual false modesty, of course, labeling my

new invention a (mere) trick in the hope of eliciting warm praise and protestations that "it's more than a trick, it's a game-changer."[6] But the hypothetical mathematical ethnographer will need to understand that not every mathematical speech act deserves to be called a trick and will want to know why.

A straightforward calculation, for example, is certainly not a trick. Nor is a syllogism, a standard estimate of magnitude, or a reference to the literature. Can I be more precise? Probably not. While capital-*M* Mathematics is neatly divided among axioms, definitions, theorems, and proofs, the mathematics of mathematicians blurs taxonomical boundaries. A mathematical trick, like a trickster, is a notorious crosser of conventional borders; a "lord of in-between" like the Yoruba trickster Eshu, "who dwells at the crossroads," a mathematical trick simultaneously disturbs the settled order and "makes this world," to quote the title of Lewis Hyde's classic study. I would suggest that a trick involves drawing attention to an intrinsic element of a mathematical situation that *appears to be, but is not, in fact, irrelevant* to the problem under consideration. Alternatively, since a trick need not be subordinated to a preexisting problem, it provides an *unexpected point of contact*, like a play on words, between two domains not previously known to be related.[7] Thus *Lieberman's trick*, the first trick I saw identified as such, roughly consists in the use of multiplication in a situation when only addition seems relevant; the *unitarian trick* of Adolf Hurwitz, Hermann Weyl, and Issai Schur introduces a measure to solve a purely algebraic problem; my trick, like other tensor-product tricks, uses the possibility of a kind of (matrix) multiplication to reveal a structure not otherwise visible. Chapter β.5 mentioned a trick connected with Cantor's theory of infinity. The reader (or listener) in each case undergoes a prototypical *Aha*! experience, the apprehension of a gestalt: the connection is obvious, but first you have to experience it.

The ambivalence of the word *trick*, whose associations include magic, prostitution, and deceit, neatly reflects the tension between satisfaction with a synoptic proof and disapproval of the shortcut that avoids the hard work, or the grappling with essential matters, without which recognition seems undeserved. As idealized by logical empiricist philosophers, Mathematics with a capital *M* is insensitive to the complex interplay of delight (a "neat trick") and disdain (a "cheap trick") that accompanies

the revelation of a new mathematical trick and constitutes a privileged moment of pleasure, precisely like that afforded by magic tricks or like García Márquez's reaction, nearly falling out of bed when he read the first sentence of Kafka's *Metamorphosis*: "I didn't know you were *allowed* to write like that."[8] The trickster is a mathematical magic realist who exclaims, "You didn't know you were allowed to fly from peak to peak; you thought you had to trek, or at least calculate, your way across the wilderness. But look: a flying carpet!" This situates the trickster at the pole opposite to the *lumberjack*, who makes his one and only appearance as an incipient mathematical archetype in one of Langlands's most oft-quoted exhortations:

> We are in a forest whose trees will not fall with a few timid hatchet blows. We have to take up the double-bitted axe and the cross-cut saw, and hope that our muscles are equal to them.[9]

The ambivalence of tricks, the sense of getting something for nothing, persists in other languages. The Dutch word *truuk* (also spelled *truc*) is "[m]ostly used in connection with magicians and card tricks . . . a 'truuk' cannot be something very serious."[10] Russian mathematicians use the word *tryuk* (*трюк*), which in other settings can mean deceit or craftiness.[11] In French a mathematical trick is called an *astuce*, whose primary association with "cleverness" or "astuteness" seems to connote approval, quite unlike the word *tour* used for magic tricks; *jouer un tour* means to "play a trick," usually not a nice one.[12]

German mathematicians nowadays often use the English word *trick* for what traditionally was, and sometimes still is, called a *Kunstgriff*; this is how one properly refers in German to the unitarian trick of Hurwitz, Weyl, and Schur. Stüttgart professor Wolfgang Rump assigned the *Kunstgriff* a legitimate role in mathematics:

> [T]ricks precede a theory, they reach into the as yet unknown, connect what is apparently separate, so that after further reflection the latter finds its natural place in the general theory and thereby becomes known.[13]

Like the African American trickster High John de Conquer, a mathematical *Kunstgriff* "mak[es] a way out of no-way."[14] But the German word has its own unsavory connotations. Arthur Schopenhauer's unpublished 1830 manuscript, entitled *Kunstgriffe*, outlines thirty-eight rhetorical

Kunstgriffe—"stratagems" in English—included in the posthumous compilation entitled *The art of being right.* It could be described as a list of dirty tricks for winning arguments or, alternatively, a training manual for the "clever talkers" of the Confucian epigraph. *Kunstgriffe* of this sort—for example, number 24, which offers advice on "stating a false syllogism"—are uniformly unwelcome in mathematics, but number four on the list may provide insight into the construction of this book and of the present chapter in particular:

> **Kunstgriff 4**
>
> If you want to draw a conclusion, you must not let it be foreseen, but you must get the premisses admitted one by one, unobserved, mingling them here and there in your talk. . . . Or, if it is doubtful whether your opponent will admit them, you must advance the premisses of these premisses. . . . In this way you conceal your game until you have obtained all the admissions that are necessary, and so reach your goal by making a circuit.[15]

Three Functions

Mathematicians used to air their ambivalent feelings about tricks in public. One finds educators opposing tricks to knowledge in 1909:

> Much time was spent in trying to find a simpler way [to solve an examination problem] until the "trick" required was found. . . . The question remains: should such questions, based upon the use of special artifices, be set in examinations . . . ? They are no test of the knowledge of candidates, and merely lead them into traps from which they emerge disheartened.[16]

As recently as 1940, the Mathematical Association of America (MAA) could publish an article by a college teacher who complained "that most of us proceed to teach certain sections of elementary mathematics in a way that discourages students by giving them the impression that excellence in mathematical science is a matter of trick methods and even legerdemain."[17] In contrast, in its advice to prospective authors of mathematical articles, the American Mathematical Society (AMS) gives tricks a positive valuation: "Omit any computation which is routine (i.e., does not depend on unexpected tricks). Merely indicate the